

MOND-like acceleration in integrable Weyl geometric gravity

*Erhard Scholz**

Oct. 15, 2015

Abstract

In this paper a Weyl geometric scalar tensor theory of gravity with scalar field ϕ and scale invariant cubic (“aquadratic”) kinetic Lagrangian is introduced. Einstein gauge (comparable to Einstein frame in Jordan-Brans-Dicke theory) is most natural for studying trajectories. In it, the Weylian scale connection induces an additional acceleration which in the weak field, static, low velocity limit acquires the deep MOND form of Milgrom/Bekenstein’s gravity. The energy momentum of ϕ leads to another add on to Newton acceleration. Both additional accelerations together imply a MOND-ian phenomenology of the model. It has unusual transition functions $\mu_w(x), \nu_w(y)$. They imply higher phantom energy density than in the case of the more common MOND models with transition functions $\mu_1(x), \mu_2(x)$. A considerable part of it is due to the scalar field’s energy density which, in our model, gives a scale and generally covariant expression for the self-energy of the gravitational field.

Contents

1	Introduction	2
2	A Weyl geometric approach to gravity	4
3	Weyl geometric scalar tensor theory (W-ST)	11
4	MOND approximation	18
5	Comparison with other MOND models	23
6	Discussion	27
7	Appendices	30

*University of Wuppertal, Department C, Mathematics, and Interdisciplinary Centre for History and Philosophy of Science; scholz@math.uni-wuppertal.de

1 Introduction

Shortly after Milgrom originally proposed his modified Newtonian dynamics, MOND, as an explanation for the observed anomalies in galaxy rotation curves, he and Bekenstein showed how a MONDian dynamics could be derived from a Lagrangian of a scalar field ϕ . It involved a kinetic term of the scalar field, proportional to $\tilde{f}(a_o^{-2}(\nabla\phi)^2)$ with a non-linear functional \tilde{f} [6].¹ A case distinction between the Newton regime and the MOND regime had to be inbuilt into the functional \tilde{f} . In the appendix of their paper they indicated how their “a-quadratic” (AQUAL) Lagrangian could be adapted to general relativity in a Jordan-Brans-Dicke (JBD) framework. This approach was the first of a collection of different attempts to cope with MOND phenomenology in general relativistic frameworks (TeVSe, Einstein aether, and others). The relativistic a-quadratic Lagrangian approach itself (“RAQUAL”) suffered from certain deficiencies noticed by the authors from the outset: gravitational waves appeared to propagate with velocity greater than that of light; gravitational lensing and cluster dynamics could not be accounted for. Moreover, the different conformal aspects in JBD theory, “Jordan frame” and “Einstein frame”, entered the analysis in a rather unclear way, typical for JBD-theory at the time.²

In the meantime it has become clear that such different, conformally related, “frames” are better analyzed in terms of integrable Weyl geometry. There they reappear as different scale gauges of the (conformal) class of pseudo-Riemannian metrics.³ But, alas, the Weyl geometric approach to gravity is not yet well known in mainstream gravity theory. Therefore this paper starts with short introductions to (integrable) Weyl geometry (section 2) and its consequences for gravity theory (section 3) in order to make it relatively self-contained. We then analyze how the original AQUAL Lagrangian can be put into a scale invariant form. Scale invariance constrains its form strongly. In its most simple form it is given by a cubic expression in the gradient of the scalar field. In Einstein gauge, the scale covariant coefficient of this term turns into a constant \tilde{a}_o which plays a role analogous to the MOND constant a_o , but is not identical with it (section 4).

The conceptual clarification achieved by this move is striking: In the weak field, static, low velocity approximation the metrical representation of the Newton potential is kept intact for the Riemannian component of the Weyl metric, while the Weylian scale connection induces an additional accel-

¹ a_o denoted the typical new constant of the MOND hypothesis, $a_o \approx \frac{1}{6}H_o[c]$, H_o the Hubble constant in time units, c the velocity of light.

²Still in later presentations Bekenstein conceived the Jordan frame as “the metric measured by rods and clocks, hence the physical metric”, while Einstein frame played the role of a “primitive metric” which governed the Einstein-Hilbert action “in order not to break violently with GR ...” [5, p. 5f.].

³[38] or [48, sec. 3].

eration for the dynamics of test bodies. It has the scale invariant form of the scalar field in Riemann gauge as its potential. The additional acceleration is part of an extended metrical theory of gravity; it needs no other structural element (section 2.4). Specifying these general considerations to the case of a scalar field with the cubic Lagrangian introduced in section 3.1 leads, in good approximation, to a MOND-like modified Poisson equation very much like in RAQUAL. But here it governs only the (“anomalous”) additional acceleration induced by the Weylian scale connection, while the Riemannian component remains governed by the ordinary Poisson equation (which will acquire an additional source term, as we shall see in a moment). Conditions for the applicability of this (MOND-) approximation are estimated. In the MOND and deep MOND regimes the condition is satisfied for star neighbourhoods; on larger scales it even may promise a better understanding of cluster dynamics (section 4.1).

A new feature arises from the evaluation of the energy-momentum tensor of the scalar field in the Weyl geometric framework. The most important contributions to the energy tensor derive from boundary terms in varying the modified Hilbert action. Here they give rise to an energy density of the scalar field, which cannot be neglected for the dynamics of the systems under study (section 4.3). They add a scalar field contribution to the right hand side of the Newtonian Poisson equation and lead to a second addition to the Newton acceleration, proportional to the MOND acceleration of the scale connection. The effect of both additions is to be equated with the empirically determined acceleration in the deep MOND regime (section 4.4). This requires the constant \tilde{a}_o to be $\frac{1}{16}a_o$. Then the weak field, static, low velocity limit of the Weyl geometric gravity theory acquires a MONDian phenomenology.

The Weyl geometric MOND model has (well-determined, not freely selectable) transition functions $\mu_w(x)$ and $\nu_w(y)$ which describe the transformation from Newton acceleration to the total modified acceleration, although *only in the upper transition regime* to the deep MOND domain (cf. appendix 7.3). To my knowledge, the resulting transition functions have not yet been considered in the literature; here they are compared with some transition functions which are in use for modelling galaxies or galaxy clusters in the astronomical literature ($\mu_1(x)$, $\mu_2(x)$ and the corresponding ν -functions). This comparison shows that the so-called “phantom” energy density is higher in the Weyl geometric model (section 5.1).

A short discussion of the outcome of our analysis follows (section 6).

2 A Weyl geometric approach to gravity

2.1 Some basics of Weyl geometry

We use Weyl geometry as our geometric framework.⁴ It combines a *conformal structure*, given by an equivalence class $\mathfrak{c} = [g]$ of pseudo-Riemannian metrics $g : ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ (in local coordinates) and a *uniquely determined affine connection* Γ (in local coordinates $\Gamma_{\mu\nu}^\nu$) with *covariant derivative* ∇ . The two constitutive elements of the structure \mathfrak{c} and ∇ (respectively Γ) satisfy the following *compatibility* condition: Any choice of g in \mathfrak{c} specifies a real valued differential 1-form φ which depends on g , in coordinates $\varphi = \varphi_\mu dx^\mu$, such that the covariant derivative of g is $\nabla_\lambda g_{\mu\nu} = -2\varphi_\lambda g_{\mu\nu}$, i.e.

$$\nabla g + 2\varphi \otimes g = 0. \quad (1)$$

In the mathematical literature a pair of data (\mathfrak{c}, ∇) satisfying (1) is called a *Weyl structure*.⁵

A change of the conformal representative

$$g \mapsto \tilde{g} = \Omega^2 g = e^{2\omega} g, \quad \omega = \ln \Omega, \quad (2)$$

with diff'ble functions Ω or ω , is accompanied by a change of the 1-form

$$\varphi \mapsto \tilde{\varphi} = \varphi - d \ln \Omega = \varphi - d\omega. \quad (3)$$

This is the local description of a *gauge transformation* for the connection φ in the trivial line bundle over spacetime of the scaling group (\mathbb{R}^+, \cdot) .

The change of the conformal representative g has a natural physical interpretation as a point dependent *change of measurement units*, of scale (or “length”) *gauge* as Weyl called it.⁶ With basic physical units expressed in terms of time as the only elementary quantity and natural constants, like in the new SI regulations, the scale change of length/time units induces a coherent rescaling of the most important basic SI units.⁷ Weyl introduced (3) as a gauge transformation of the scale connection long before the general theory of connections in principal fibre bundles was developed, or the SI headed towards universal natural units of measurements [60]. In his view the primary data of the generalized geometrical structure were given by pairs (g, φ) under the equivalence $((2), (3))$. Accordingly we call the equivalence class

$$[(g, \varphi)] \quad \text{a Weyl(ian) metric.} \quad (4)$$

Any specific choice of (g, φ) is a (scale) *gauge* of the Weylian metric, g its *Riemannian component* and φ the corresponding *scale connection*.

⁴For more details see, among many others, [1, 7, 36, 37, 47] from the point of view of physics, for a differential geometric perspective [21, 27, 24].

⁵[27, 9, 34, 24].

⁶Compare with Brans/Dicke’s view, most clearly expressed in [16, p. 2163].

⁷[8], (www.bipm.org/en/si/new-si/)

Weyl geometry is closely related to conformal geometry; its main difference is the *unique* determination of an *invariant affine connection* (and with it a covariant derivative). For any choice (g, φ) , the invariant affine connection may be expressed in terms of the (scale dependent) Levi-Civita connection ${}_g\Gamma_{\nu\lambda}^\mu$ of the Riemannian component g and an additional term ${}_\varphi\Gamma_{\nu\lambda}^\mu$ depending on the scale connection:

$$\Gamma_{\nu\lambda}^\mu = {}_g\Gamma_{\nu\lambda}^\mu + {}_\varphi\Gamma_{\nu\lambda}^\mu, \quad {}_\varphi\Gamma_{\nu\lambda}^\mu = \delta_\nu^\mu \varphi_\lambda + \delta_\lambda^\mu \varphi_\nu - g_{\nu\lambda} \varphi^\mu. \quad (5)$$

The Riemann and Ricci tensors $Riem$, $Ricc$ of the affine connection are *invariant* under scale change although it is possible, and often important, to express them in terms of the scale dependent quantities g and φ in the form $Riem = {}_gRiem + {}_\varphi Riem$, with ${}_gRiem$ the Riemannian curvature derived from the Levi-Civita connection of g and ${}_\varphi Riem$ the correction term derived from the scale connection φ ; similarly $Ricc = {}_gRicc + {}_\varphi Ricc$.⁸

The Weyl geometric scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$ is not scale invariant but scales with $g^{\mu\nu}$ (weight -2 , cf. below). It is composed from the scalar curvature of the Riemannian component ${}_gR$ and a term collecting the influence of the scale connection ${}_\varphi R$

$$\begin{aligned} R &= {}_gR + {}_\varphi R \\ {}_\varphi R &= -(n-1)(n-2)\varphi_\lambda \varphi^\lambda - 2(n-1)g\nabla_\lambda \varphi^\lambda \\ &= -6\varphi_\lambda \varphi^\lambda - 6g\nabla_\lambda \varphi^\lambda \quad \text{in dimension } n = 4. \end{aligned} \quad (6)$$

Of course, the scale connection has a curvature f of its own. Because the commutative scale group it is simply the exterior derivative

$$f = d\varphi \quad (\text{scale curvature}). \quad (7)$$

If it vanishes, $d\varphi = 0$, there is a scale choice of the Weylian metric, $(\tilde{g}, 0)$, in which the scale connection vanishes (*integrable Weyl geometry*). Then the Weyl metric *looks* Riemannian in this gauge; but it would be a mistake to identify it with the Riemannian metric g because the underlying scale covariance group is *not reduced* to the identity. Even in the case of an *integrable Weyl geometry* the group of *geometrical automorphisms* contains the *conformal* transformations. It is important to keep this (simple) observation in mind for the study of scalar tensor theory of gravity in the Weyl geometric framework.⁹

Some geometrical and many physical quantities are given by fields X which transform under rescaling. Mathematically speaking, such fields live

⁸For explicit formulas see the literature given in fn. 4

⁹In his reflections on the quantization of gravity 't Hooft considers "local conformal symmetry" as an exact symmetry, although explicitly avoiding to make use of the Weyl geometric framework [56, fn. 2]. Perhaps it would be helpful to give up this methodological restriction.

(i.e. have values) in bundles over spacetime with non-trivial representation of the scale group. A field X transforming by $\tilde{X} = \Omega^k X = e^{k\omega} X$ under (2) is known as *scale covariant* field of Weyl *weight* k . For geometrical reasons we work with length/time weights, inverse to energy weights preferred in high energy physics by obvious reasons. The *scale covariant derivative* D of such a field X responds to the non-trivial weight; it is given by

$$DX := \nabla X + w(X)\varphi \otimes X . \quad (8)$$

We now see that the compatibility (1) means $Dg = 0$, i.e. the *scale covariant derivative of the metric vanishes* – a Weyl geometric analogue of the metricity condition for the Levi-Civita connection in Riemannian geometry.

In addition to the notations ∇ for *scale invariant* covariant differentiation and D for *scale covariant* differentiation of fields we shall use the notation ${}_g\nabla$ for the scale dependent differentiation with regard to the Levi-Civita connection of the Riemannian component g of a Weyl metric given in gauge (g, φ) .

Weyl geometry connects to physics via different routes. Leaving aside Weyl's own idea of a geometrically unified theory of electromagnetism and gravity, two different research programs developed in the second half of the 20th century. The first one in the theory of gravity (with links to elementary particle physics and cosmology) characterized by a gravitational scalar field non-minimally coupled to the scalar curvature, similar to Jordan-Brans-Dicke theory (going back to M. Omote and P.A.M. Dirac in the early 1970); the second one arising from a Weyl geometric re-reading of Bohmian quantum mechanics with a scale covariant scalar field in the role of a generalized quantum potential (opened by E. Santamato in the 1980s).¹⁰ In recent years the gravitational scalar field approach has been taken up in the simplified form of *integrable* Weyl geometry. Our investigation is part of this research tradition.

2.2 Weyl geometry as a framework for gravity

Lagrangians of field theories in the Weyl geometric framework have to be invariant under scale transformation (conformal invariance). It is advisable to express them in terms of the scale co- or invariant expressions outlined above. Weyl himself worked with quadratic expressions in the curvature to get scale invariant Lagrangians. A similar approach is still used in conformal theories of gravity.¹¹ But roughly a decade after the advent of Brans-Dicke theory several authors, beginning with M. Omote and P.A.M. Dirac, formulated a Weyl geometric version of a scalar field ϕ of weight $w(\phi) = -1$ non-minimally coupled to Weylian scalar curvature R , with a *Hilbert-Weyl*

¹⁰For the quantum potential approach see, among many, [43, 13, 14, 51, 11].

¹¹ [29]

term $L_{HW} = |\phi|^2 R$.¹² Originally the Weylian scale connection was treated as a dynamical field with a Yang-Mills like Lagrange term for φ .¹³

It was soon realized that such a field would have a boson close to the Planck scale. Some authors speculated that the scalar field might arise as an order parameter of a boson condensate.¹⁴ In such a case, the low energy effective Lagrangian does not attribute an independent dynamical role to the scale connection because the scale curvature vanishes for low energies.¹⁵ The only additional dynamical effect of the field theoretic extension is due to the scalar field. A *geometrical* role of the scale connection remains even in this case of an integrable Weyl geometry. All this is consistent with the outcome of Ehlers/Pirani/Schild's analysis on the foundational role of Weyl geometry, and the succeeding investigations of Audretsch/Gähler/Straumann.¹⁶

We arrive at a scalar tensor theory of gravity (and other fields) with a Lagrangian of the general form

$$\begin{aligned} L &= \alpha \phi^2 R + \dots \\ \mathcal{L} &= L \sqrt{|g|}, \quad |g| = |\det g|, \end{aligned} \tag{9}$$

where the dots indicate scalar field, matter and interaction terms. Obviously (9) is very close to Jordan-Brans-Dicke theory (JBD). The crucial difference is that in our case the scalar curvature R and all dynamical terms are consistently expressed in Weyl geometric scale covariant form and the Lagrangian remains scale (conformally) invariant for any α , not only for $\alpha = \frac{1}{6}$. Scale covariance has not to be broken by hand. There are no “two” (or even more) “metrics” involved. The notorious question of “physicality” of frames in JBD theory is brought into a different (and clarifying) light.¹⁷ In short, the Weyl geometric framework brings in more clarity of concepts and simplifies calculations.

2.3 Scale invariant observables and two distinguished gauges

It is clear how to extract *scale invariant observable magnitudes* \check{X} from a scale covariant field X of weight $w(X) = k$. One only has to form the proportion with regard to the appropriate power of the scalar field's norm

$$\check{X} := X / |\phi|^{-k} = X |\phi|^k; \tag{10}$$

¹²[31, 17, 32, 57, 58, 25].

¹³Dirac continued to interpret φ as electromagnetic potential, while the Japanese physicists hoped for a new insight into nuclear fields.

¹⁴[25, 52, 12, 26].

¹⁵Curvature effects can be seen only at lengths/energies close to the Planck scale.

¹⁶[19] show that the causal structure and a compatible non-chronometric inertial structure (mathematically a conformal and a compatible path structure) uniquely specify a Weylian metric. [4] have shown that, in the WKB approximation, the streamlines of a Klein-Gordon field approximate the geodesics of the Weyl metric if and only if the scale curvature vanishes.

¹⁷[38, 37, 35, 39, 2, 48].

then clearly $w(\check{X}) = 0$.

Scale invariant magnitudes \check{X} are directly indicated, up to a globally constant factor in *scalar field gauge*, i.e., the gauge in which

$$|\phi| \doteq \text{const} =: \phi_o, \quad (11)$$

where \doteq indicates an *equality which holds in a specified gauge only* (here scalar field gauge). In [57] ϕ is therefore called a “measuring field”. In our context ϕ will be strictly positive real valued; thus we can omit the norm signs in the expressions above. By consistency considerations with Einstein gravity we have to postulate that in scalar field gauge

$$\alpha\phi^2 \doteq \alpha\phi_o^2 = (16\pi G)^{-1}, \quad (12)$$

Scalar field gauge with (12) will be called and denoted by

$$(\hat{g}, \hat{\varphi}) \quad \textit{Einstein (- scalar field) gauge}. \quad (13)$$

Once the context is clear, the hats may be (and will be) omitted.

In integrable Weyl geometry there is another distinguished gauge of the form $(\tilde{g}, 0)$ in which the scale connection vanishes. By obvious reasons it is called

$$(\tilde{g}, 0) \quad \textit{Riemann gauge} \quad (14)$$

(“Jordan frame” in JBD theory). Writing the scalar field in Riemann gauge $\tilde{\phi}$ in exponential form, $\tilde{\phi} = e^{\tilde{\omega}}$, turns its exponent

$$\tilde{\omega} := \ln \tilde{\phi} \quad (15)$$

into a *scale invariant* expression for the scalar field. (Further below, we shall omit the tilde sign, if the context makes clear that the scale invariant exponent is meant.) The scale connection $\varphi = \hat{\varphi}$ in scalar field gauge is then

$$\hat{\varphi} = -d\tilde{\omega}, \quad (16)$$

because $\Omega = \tilde{\phi}$ is the rescaling function from Riemann to scalar field gauge.

Riemann gauge and scalar field/Einstein gauge are the most important gauges in Weyl geometric scalar field theory. In the first one, the affine connection is identical to the Levi-Civita connection of the Riemannian component \tilde{g} .¹⁸ In the second one, the coefficient of scalar curvature is consistent with Einstein gravity and the scale invariant observables are directly indicated by the field quantities without further calculation (up to a global constant). We may expect, or postulate, that clock measurements are indicated by quantities in this gauge.¹⁹ Thus both gauges have their mathematical *and physical* values and vices; both indicate some physically important feature most directly, while others have to be extracted by additional calculations. Both are equivalent mathematically.

¹⁸Some authors in the JBD approach consider this as the criterion for the “physical” gauge [5].

¹⁹For a possible physical reason, mediated by a link to the Higgs field, see [49].

2.4 Inertio-gravitational, conformal, and chronometric structures

Scale invariant geodesics are the autoparallels of the scale invariant derivative, i.e. paths $\gamma(t)$ satisfying

$$\nabla_{\dot{\gamma}}(\dot{\gamma}) = 0 \quad \longleftrightarrow \quad \ddot{\gamma}^\lambda + \Gamma_{\mu\nu}^\lambda \dot{\gamma}^\mu \dot{\gamma}^\nu = 0. \quad (17)$$

The corresponding *scale covariant geodesics* arise from (17) by reparametrizing these paths to unit length in any gauge. Their vector fields $u(t) = \dot{\gamma}(t)$, defined along every path, are of weight $w(u) = -1$; then we have $g(u, u) = \pm 1$ independent of the scale gauge. They are given by

$$D_u u = \nabla_u u - \varphi(u)u = 0 \quad \longleftrightarrow \quad \dot{u}^\lambda + \Gamma_{\mu\nu}^\lambda u^\mu u^\nu - \varphi_\mu u^\mu u^\lambda = 0. \quad (18)$$

The autoparallels of (18) differ from Weyl's scale invariant geodesics (17) by parametrization only and constitute a class of *covariantly parametrized geodesics*.²⁰ They are the autoparallels of a projectively related class $[\tilde{\Gamma}(\varphi)]$ of affine connections $\tilde{\Gamma}(\varphi)$ depending on the gauge (g, φ) :

$$\tilde{\Gamma}(\varphi)_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \frac{1}{2}(\delta_\nu^\mu \varphi_\kappa + \delta_\kappa^\mu \varphi_\nu) \quad (19)$$

Here the additional term arising from scale covariant derivation of weight -1 has been underlined. The class $[\tilde{\Gamma}]$ characterizes a *projective* path structure $[\gamma]$ with paths given by (18).²¹

According to the analysis of Ehlers/Pirani/Schild the projective and the conformal structure \mathfrak{c} specify the affine connection and its covariant derivative ∇ uniquely. As also the Weyl structure specifies the projective structure we have three equivalent characterizations of a Weyl geometry:

$$(\mathfrak{c}, [\tilde{\Gamma}]) \quad \longleftrightarrow \quad (\mathfrak{c}, \nabla) \quad \longleftrightarrow \quad [(g, \varphi)], \quad (20)$$

with $[(g, \varphi)]$ a Weylian metric in the sense of (4). Each of them defines an *inertio-gravitational* structure in the sense of Weyl while the chronometry is still undetermined up to a point dependent scale factor.

As shown in section 2.3, a scale covariant scalar field ϕ as in section 2.2 specifies a *chronometry*. A Weylian metric plus a scalar field $[(g, \varphi, \phi)]$ thus determine a full-fledged *spacetime structure* in the sense of [53]. Remember that in the case of an integrable Weyl structure φ and ϕ are not dynamically independent but determine each other mutually. Any Weyl geometric

²⁰More generally, a path γ in a Weylian spacetime manifold M is called *scale covariantly parametrized* of weight -1 , if to any scale choice (g, φ, ϕ) a parametrization $\gamma: \mathbb{R} \rightarrow M$ is given, which changes under rescaling of the metric in such a way that $g(\gamma(\tau), \gamma(\tau))$ is independent of the gauge.

²¹That (17) and (18) characterize the same path structure can be verified by the criterion of *projective equivalence* for two connections $\Gamma, \tilde{\Gamma}$, which is $(\tilde{\Gamma} - \Gamma)_{\nu\kappa}^\mu X^\nu X^\kappa \sim X^\mu$ for any vector field X .

scalar field theory contains point dependent rescaling as a subgroup of its automorphisms. The choice of Einstein - scalar field gauge allows to specify the chronometric structure in an adapted way but does not reduce the group of automorphisms.

2.5 Additional acceleration induced by the scale connection

Free fall of test particles in Weyl geometric gravity follows scale covariant geodesics $\gamma(\tau)$ of weight $w(\dot{\gamma}) = -1$. Slow (non-relativistic) motions are described by a differential equation formally identical to the one in Einstein gravity, but with scale covariant derivatives of the Weyl geometric affine connection rather than that of the (Riemannian) Levi-Civita one.

Coordinate acceleration a with regard to proper time t for a low velocity motion parametrized by $x(t)$ is given (analogous to Einstein gravity) by²²

$$a^j = \frac{d^2 x^j}{dt^2} \approx -\Gamma_{oo}^j. \quad (21)$$

Because of (5) the total acceleration decomposes into

$$a^j = -g\Gamma_{oo}^j - \varphi\Gamma_{\nu\lambda}^j = a_R^j + a_\varphi^j, \quad (22)$$

where $a_R^j = -g\Gamma_{oo}^j$ is the Riemannian component of the acceleration known from Einstein gravity, and $a_\varphi^j = -\varphi\Gamma_{oo}^j$ an *additional acceleration* due to the Weylian scale connection.

For a (diagonalized) weak field approximation in Einstein gauge,

$$g_{\mu\nu} \doteq \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \quad (23)$$

with $\eta = \epsilon_{sig} \text{diag}(-1, +1, +1, +1)$, the Riemann-Einstein component is standard:

$$a_R^j = -g\Gamma_{oo}^j \approx \frac{1}{2}\eta^{jj}\partial_j h_{oo}, \quad (24)$$

neglecting 2-nd order terms in h . In the light of (5) and (100) the additional Weylian component becomes

$$a_\varphi^j \doteq g_{oo}\varphi^j \doteq g_{oo}g^{jj}\partial_j \tilde{\omega} \approx -\partial_j \tilde{\omega} \approx \varphi_j, \quad (25)$$

This shows that in the static weak field, low velocity case and in Einstein gauge the Weylian *scale connection* represents an *additional acceleration*.

Because of (16) the *invariant* form of the *scalar field* $\tilde{\omega}$ can be identified with the *potential of the additional acceleration* (weak field approximation, Einstein gauge), analogous to Einstein's identification of the Newton potential with a metrical perturbation, $\Phi_N := -\frac{1}{2}\epsilon_{sig}h_{oo}$:

$$a_R \approx -\nabla\Phi_N = -\frac{1}{2}\epsilon_{sig}\nabla h_{oo} \quad (26)$$

$$a_\varphi \approx -\nabla\tilde{\omega} \quad (27)$$

²²[59, pp. 213ff.] or, for Weyl geometry, [45, eq.(60)].

3 Weyl geometric scalar tensor theory (W-ST)

3.1 ... with a cubic scalar field Lagrangian

Our Lagrangian density $\mathcal{L} = L\sqrt{|g|}$ contains a Hilbert-Weyl term L_{HW} , a dynamical term L_ϕ and a potential term L_{V4} for the scalar field and a matter term L_m , all of them of weight -4 :

$$L = L_{HW} + L_{V4} + L_\phi + L_m$$

We assume a classical matter term with $w(L_m) = -4$ comparable to the matter terms of the standard model fields, for which test particles follow the *Weyl geometric path structure*. The postulate is strongly supported by the analysis of the stream lines of a Klein-Gordon field (in WKB approximation) [4], if one assumes a structure-conserving transition from the quantum world to classical particle motion after decoherence. It can be understood as a compatibility criterion of the matter Lagrangian with the EPS axioms for a generalized theory of gravity (Ehlers/Pirani/Schild).²³

For covering both signature choices for g , preferentially used in gravity theory or in elementary particle physics, we introduce

$$\epsilon_{sig} = \begin{cases} +1 & \text{if } \text{sign}(g) = (3, 1) \sim (- +++) \\ -1 & \text{if } \text{sign}(g) = (1, 3) \sim (+ ---) \end{cases} \quad (28)$$

and a modified Hilbert term typical for scalar-tensor theories of gravity, adapted to the Weyl geometric framework:

$$L_{HW} = \frac{\epsilon_{sig}}{2}(\xi\phi)^2 R \quad \text{Hilbert-Weyl term,} \quad (29)$$

$$L_{V4} = -\frac{\lambda}{4}\phi^4 \quad \text{quartic potential term of } \phi, \quad (30)$$

with constants ξ, λ to be interpreted later. R is the Weyl geometric scalar curvature, scale covariant of weight $w(R) = -2$. The coefficient ξ has to be fixed such that in scalar field/Einstein gauge $(\xi\phi)^{-2} \doteq 8\pi G$. So far all *Weyl geometric scalar tensor theories* of gravity (W-ST) coincide.

Usually the dynamical term L_ϕ of the scalar field is quadratic in its (scale covariant) gradient, i.e. proportional to $(D\phi)^2 = D_\nu\phi D^\nu\phi$. In order to adapt to our form of the Hilbert term we write it in the form

$$L_{\phi 2} = \epsilon_{sig} \frac{\alpha}{2} \xi^2 |D\phi|^2 = \epsilon_{sig} \frac{\alpha}{2} \xi^2 D_\nu\phi D^\nu\phi. \quad (31)$$

²³ This assumption deserves further investigation. It can be stated as an action principle for point particles with the scale invariant action: $S_{pp} = \int \phi_{comp} \sqrt{g(\dot{\gamma}\dot{\gamma})} d\tau$ (with γ timelike curves parametrized by τ , ϕ_{comp} the “compensating field” like in appendix 7.1); but the question of consistency or derivability would still persist. In [3] it is derived for a weak extension of Einstein gravity, rewritten scale covariantly using Weyl geometry (by means of the contracted Bianchi identity applied to the energy-momentum of dust-like matter, like in ordinary Einstein gravity). This approach might be generalizable. The condition of *EPS compatibility* is analyzed in great generality in [15].

But here we want to reconsider the alternative of an *aquadratic* Lagrangian proposed by Bekenstein/Milgrom for reproducing the non-linear Poisson equation of the MOND phenomenology in the static weak field limit,²⁴

$$(8\pi G)^{-1} c^{-2} f(c^2 (\nabla\phi)^2), \quad (32)$$

where f is a non-linear function and the constant c has “dimensions of length introduced for dimensional consistency” [5, p. 6]. Bekenstein’s f could be chosen among a large class of functions (it is not “not known *a priori*”) and is functionally related to the MOND specific transition function $\mu(x)$ from the Newton regime to the deep MOND domain. That implies the asymptotic condition

$$f(y) \rightarrow y^{\frac{3}{2}} \quad (\text{up to a constant factor}) \quad \text{for} \quad y \rightarrow 1. \quad (33)$$

Assimilating (32) to our context, f will be strongly constrained by the total weight condition $w(L_\phi) = -4$ and the asymptotic condition (33). The simplest non-quadratic form is $f(y) = y^{\frac{3}{2}}$ itself (for $y \geq 0$), with a reduction of the exponent of the factor c^{-2} in front of f in Bekenstein’s Lagrangian to -1 .

For achieving scale invariance of \mathcal{L}_ϕ we set

$$L_{\phi 3} = \frac{2}{3} \xi^2 \eta \phi^{-2} \|D\phi\|^3 \quad (34)$$

and add it to $L_{\phi 2}$ for the kinetic term of ϕ . $D\phi$ denotes the *scale covariant gradient* of ϕ with components $D^\nu \phi$; the “norm” $\|X\|$ of a 4-vector $X = (X^\nu)$ is to be read as

$$\|X\| = \text{Re} (\epsilon_{sig} X^\nu X_\nu)^{\frac{1}{2}}. \quad (35)$$

For spacelike vectors it is the usual norm, for timelike vectors it is zero.²⁵ The coefficient η allows to adapt the model to Bekenstein/Milgrom’s value of their constant a_o . The factor $\frac{2}{3}$ is for convenience. The scale weight of $\|D\phi\|$ is -2 , thus $w(L_\phi) = 2 - 3 \cdot 2 = -4$. The condition of scale invariance for \mathfrak{L}_ϕ constrains Bekenstein/Milgrom’s f considerably.

Adapted to (35) there is now also the possibility to consider

$$L_{\phi 2} = \epsilon_{sig} \frac{\alpha}{2} \xi^2 \|D\phi\|^2 \quad (36)$$

as an alternative for the quadratic term, while in any case

$$L_\phi = L_{\phi 2} + L_{\phi 3}. \quad (37)$$

²⁴[6]

²⁵As an alternative convention one might consider $\|X\| = |X^\nu X_\nu|^{\frac{1}{2}}$. Consequences of this alternative convention, e.g. for cosmological solutions or propagation of perturbations, are still to be explored.

Although $L_{\phi 2}$ is scale covariant of weight -4 for any choice of α , the specific choice $\alpha = 6$ leads to the effect that in vacuum the scalar field equation derived from the quadratic term $L_{\phi 2}$ alone reduces to the trace of the Einstein equation. This property will allow to simplify the total scalar field equation of our L_{ϕ} considerably (subsection 3.3).

The gradient of the scalar field in terms of its invariant form $\tilde{\omega}$ (15) is $D^{\nu}\phi = \phi \partial^{\nu}\tilde{\omega}$ (appendix 7.1, eq. (101)). Thus the scalar field Lagrangian can also be written with

$$L_{\phi 3} = \frac{2}{3}\xi^2\eta\phi \|\nabla\tilde{\omega}\|^3, \quad (38)$$

with $\nabla\tilde{\omega}$ the gradient of $\tilde{\omega}$. In Einstein gauge (41), with constant value ϕ_o of the scalar field, we introduce the new constant $\eta^{-1}\phi_o = \tilde{a}_o$.²⁶ Below it will turn out that this will be realized with $\tilde{a}_o = \frac{a_o}{16}$. $L_{\phi 3}$ is *cubic* in the gradient of the scale invariant scalar field rather than quadratic (and of the correct weight because of $w(\|\nabla\tilde{\omega}\|) = -1$). In the following we shall omit the tilde and simply write ω for the latter.

3.2 Compatibility conditions

Our Lagrangian is consistent with Einstein gravity if in scalar field gauge

$$\xi\phi_o \doteq (8\pi G)^{-\frac{1}{2}} = E_{pl} \leftrightarrow L_{pl}^{-1}, \quad (39)$$

where E_{pl} , L_{pl} denote the *reduced* Planck energy and Planck length, respectively. They are normed such that

$$E_{pl} L_{pl}^{-1} = (8\pi G)^{-1}. \quad (40)$$

Obvious factors c and \hbar are omitted. Einstein gravity arises if in scalar field gauge $\varphi \rightarrow 0$.

Let us introduce the notation

$$\tilde{a}_o = \eta^{-1}\phi_o \quad (41)$$

with ϕ_o as in (11). The constant \tilde{a}_o plays a role analogous to the MOND acceleration $a_o \approx \frac{1}{6}H$, where H denotes the Hubble parameter ($H = H_o \leftrightarrow H_1$). Below we find that we have to set $\tilde{a}_o \approx \frac{a_o}{16}$ if we want to link up to Bekenstein/Milgrom's RAQUAL with the usual MOND acceleration. Einstein gravity is (precisely) contained in our approach as the special case with $\omega = \text{const.}$ Then Riemann gauge and Einstein gauge coincide and the scalar field is dynamically inert.²⁷ In the following we shall understand by *Einstein gauge* the scalar field gauge with (39) and (41).

²⁶Then $\xi^2\eta\phi = (\xi\phi)^2(\eta^{-1}\phi)^{-1} \doteq (8\pi G)^{-1}\tilde{a}_o^{-1}$ (in Einstein gauge).

²⁷[49, sect.3], [39].

ϕ_o^{-1} stands between the largest and smallest physically conceivable length units in the universe \tilde{a}_o^{-1} and L_{pl} ; or reciprocally:

$$\tilde{a}_o \xrightarrow{\cdot\eta} \phi_o \xrightarrow{\cdot\xi} E_{pl} \leftrightarrow L_{pl}^{-1}$$

The product of our typical coefficients is the ratio of these extremal quantities:

$$\eta \cdot \xi = \frac{E_{pl}}{\tilde{a}_o} = \frac{\tilde{a}_o^{-1}}{L_{pl}} \sim 10^{63} \quad (42)$$

It seems natural (although not necessary) to assume ξ and η to be at roughly comparable orders of magnitude. Then ϕ_o lies close to the geometrical mean between the extremes \tilde{a}_o and E_{pl} :

$$|\phi_o| \sim 10^{-4} \text{ eV} \quad \text{respectively} \quad 10 \text{ cm}^{-1} \quad (43)$$

3.3 Dynamical equations

In integrable Weyl geometric structures the scale covariant variation with regard to $\delta g^{\mu\nu}$ leads to the Euler-Lagrange equation

$$\frac{\delta \mathfrak{L}}{\delta g^{\mu\nu}} = \frac{\partial \mathfrak{L}}{\partial g^{\mu\nu}} - D_\lambda \frac{\partial \mathfrak{L}}{\partial (\partial D_\lambda g^{\mu\nu})}$$

with $D_\lambda g^{\mu\nu} = \partial_\lambda g^{\mu\nu} - 2\varphi_\lambda g^{\mu\nu}$ [22, p. 526]. Because of $D_\lambda g^{\mu\nu} \doteq \partial_\lambda g^{\mu\nu}$ (in Riemann gauge) the variation is most simple in Riemann gauge and close to the usual calculations. The result can be generalized to other gauges by scale transformation.²⁸

The variation with regard to $\delta g^{\mu\nu}$ leads to boundary contributions from the Hilbert-Weyl term, which vanish for a constant coefficient like in Einstein gravity.²⁹

$$\frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_{HW}}{\delta g^{\mu\nu}} = \frac{\epsilon_{sig}}{2} \xi^2 \left(\phi^2 (Ric - \frac{R}{2} g)_{\mu\nu} - D_{(\mu} D_{\nu)} \phi^2 + D^\lambda D_\lambda \phi^2 g_{\mu\nu} \right) \quad (44)$$

Here Ric and R are the Weyl geometric Ricci tensor and scalar curvature respectively. The last two terms on the r.h.s. result from the boundary contributions of partial integration. Remember that D_μ denotes the scale covariant derivative of Weyl geometry, depending on the scale weight $w = w(X)$ of a field X (8).

The variation of the other terms is straight-forward. The energy-momentum tensor of matter is defined as usual:

$$T_{\mu\nu}^{(m)} := -\epsilon_{sig} 2 \frac{1}{\sqrt{|g|}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} \quad (45)$$

²⁸For the variation in general, not necessarily integrable, Weyl geometric structures see [55, pp. 98–101].

²⁹[7, pp. 96ff.], [23, pp. 40ff.], [55, pp. 64ff.], [18, p. 1032f.] – the boundary terms lead to the “improved” energy-momentum tensor of the scalar field in the sense of [10].

The variation of \mathcal{L}_ϕ gives a peculiar energy-momentum contribution from the scalar field to the r.h.s. (see below, (47), (48)).

We arrive at the *scale invariant Einstein equation*,

$$Ric - \frac{R}{2}g = (\xi\phi)^{-2} T^{(m)} + \Theta. \quad (46)$$

The r.h.s. consists of the energy-momentum of matter $T^{(m)}$ and the energy tensor of the scalar field Θ (up to the constant $8\pi G$ in Einstein gauge). Θ decomposes into a term (I) manifestly proportional to the Riemannian component of the metric g and an additional one (II), $\Theta = \Theta^{(I)} + \Theta^{(II)}$, such that

$$\Theta^{(I)} = \phi^{-2} \left(-D_\lambda D^\lambda \phi^2 + \epsilon_{sig} \xi^{-2} (L_{V4} + L_\phi) \right) g, \quad (47)$$

$$\Theta_{\mu\nu}^{(II)} = \phi^{-2} \left(D_\mu D_\nu \phi^2 - 2\epsilon_{sig} \xi^{-2} \frac{\partial L_\phi}{\partial g^{\mu\nu}} \right). \quad (48)$$

The contribution $\epsilon_{sig}(\xi\phi)^{-2} L_{V4} g = -\epsilon_{sig} \frac{\lambda}{4} \xi^{-2} \phi^2 g$ in (47) is a scale covariant version of the “cosmological constant” term Λg ; here

$$\Lambda = \frac{\lambda}{4} \xi^{-2} \phi^2 \quad (\text{variable}), \quad \Lambda \doteq \frac{\lambda}{4} \xi^{-2} \phi_o^2 \quad (\text{constant in Einstein gauge}). \quad (49)$$

For the variation $\delta\omega$ with regard to the scale invariant form of the scalar field ω one uses (99) (valid in any gauge) and finds

$$\frac{\partial}{\partial\omega} \phi = \frac{\partial}{\omega} e^{\omega+f} \varphi = e^{\omega+f} \varphi = \phi, . \quad (50)$$

On the other hand

$$\frac{\partial}{\partial(\partial_\nu\omega)} \|\nabla\omega\|^3 = 3\epsilon_{sig} \|\nabla\omega\| \partial^\nu\omega \quad (51)$$

for $\nabla\omega$ spacelike; otherwise it is 0.

Let us introduce the *scale covariant* (non-linear) *Milgrom operator* defined by

$$\square_M \omega = \epsilon_{sig} D_\nu (\|\nabla\omega\| \partial^\nu\omega), \quad (52)$$

and \square , the *scale covariant d'Alembert operator* for a (scale covariant) scalar field X , while ${}_g\square$ is the covariant d'Alembert operator of the Riemannian metric in any gauge:

$$\square X = -\epsilon_{sig} D_\nu D^\nu X \quad (53)$$

$${}_g\square X = -\epsilon_{sig} {}_g\nabla_\nu \partial^\nu X = -\frac{\epsilon_{sig}}{\sqrt{|g|}} \partial_\nu \left(\sqrt{|g|} X^\nu \right). \quad (54)$$

According to appendix 7.2, equ. (105) the scale covariant Euler-Lagrange equation is, for D_ϕ spacelike,

$$2L_{HW} + 4L_{V4} - 2L_{\phi 3} + \alpha \xi^2 \phi \square \phi - 2(\xi \phi)^2 (\eta^{-1} \phi)^{-1} \square_M \omega = 0.$$

For $\alpha = 6$ (and spacelike gradient D_ϕ of the scalar field) subtraction of the trace of the Einstein equation simplifies the equation to:

$$\square_M \omega = \frac{1}{2} (\xi \phi)^{-2} (\eta^{-1} \phi) \left(-\epsilon_{sig} \text{tr} T^{(m)} - 3L_{\phi 3} \right) \quad (55)$$

Both sides are of weight -3 , $\text{tr} T^{(m)}$ denotes the trace of the matter tensor. For D_ϕ timelike or null and the choice (36) for $L_{\phi 2}$ the scalar field equation reduces to the potential condition

$$L_{HW} + 2L_{V4} = 0. \quad (56)$$

In Einstein gauge (55) becomes

$$\square_M \omega \doteq 4\pi G \tilde{a}_o \text{tr} T^{(m)} - \|\nabla \omega\|^3.$$

We have to compose the scale covariant Milgrom operator from its Riemannian part and the scale connection component, $\square_M \omega = {}_g \square_M \omega + {}_\varphi \square_M \omega$, with the *covariant Milgrom operator* of Riemannian geometry defined by

$${}_g \square_M \omega = \epsilon_{sig} g \nabla_\nu (\|\nabla \omega\| \partial^\nu \omega) = \epsilon_{sig} (\partial_\nu \|\nabla \omega\| \partial^\nu \omega + \|\nabla \omega\| g \nabla_\nu \partial^\nu \omega). \quad (57)$$

Because of $w(\|\nabla \omega\| \partial^\nu \omega) = -3$ we find

$${}_\varphi \square_M \omega = \epsilon_{sig} \left(\varphi^\nu_{\nu\lambda} \|\nabla \omega\| \partial^\lambda \omega - 3\varphi_\nu \|\nabla \omega\| \partial^\nu \omega \right) = \epsilon_{sig} \varphi_\nu \|\nabla \omega\| \partial^\nu \omega.$$

In Einstein gauge ${}_\varphi \square_M \omega \doteq -\|\nabla \omega\|^3$. For a fluid with matter density ρ_m and pressure p_m , equation (55) in Einstein gauge finally simplifies to

$${}_g \square_M \omega \doteq 4\pi G \tilde{a}_o \text{tr} T^{(m)} \doteq 4\pi G \tilde{a}_o (\rho_m - 3p_m) \quad (58)$$

By obvious reasons (58) will be called the *covariant Milgrom equation*.

The Einstein equation (46) and the scalar field equation (55), respectively (58), constitute an interdependent system of differential equations. We shall study it in the following section under simplifying conditions: a static weak field case and a cosmological limit.³⁰ Before we do so, we want to point out that the Schwarzschild-de Sitter solution is a special (point symmetric) vacuum solution of (46), (55) with a trivial scalar field (constant in Riemann gauge).

³⁰In previous preprints of this paper the simplicity of (58) could not be achieved because no $L_{\phi 2}$ term ($\alpha = 6$) was included.

3.4 Schwarzschild-de Sitter solution

Our first example deals with a Weyl geometrically degenerate case with Riemann gauge ($g, \varphi \doteq 0$) identical to Einstein (scalar field gauge), $\phi \doteq \phi_o = \text{const.}$ Here g denotes the Schwarzschild-de Sitter metric of signature $(-+++)$:

$$ds^2 = -\left(1 - \frac{2M}{r} - \kappa r^2\right)dt^2 + \left(1 - \frac{2M}{r} - \kappa r^2\right)^{-1}dr^2 + r^2(dx_2^2 + \sin^2 x_2 dx_3^2) \quad (59)$$

Then $\square_M \omega = 0$, and (58) is trivially satisfied in the vacuum.

The Ricci and scalar curvatures are $Ric = 3\kappa g$, $R \doteq 12\kappa$. We calculate in scalar field gauge, while suppressing the dot of \doteq here. The l.h.s. of our Einstein equation is familiar,

$$Ric - \frac{R}{2}g = -3\kappa g.$$

In vacuum the r.h.s. of the Einstein equation (47, 48) simplifies to the quartic term (“cosmological constant”) of the scalar field potential (49):

$$\Theta = \Theta^{(I)} = -\frac{\lambda}{4}\xi^{-2}\phi_o^2 g = -\frac{\lambda}{4}\beta^2 \tilde{a}_o^2 g$$

where β denotes the ratio $\beta = \eta \xi^{-1}$ which, according to (43) is no large number. Then (46) is satisfied for

$$3\kappa = \frac{\lambda}{4}\beta^2 \tilde{a}_o^2$$

Below we shall find $\tilde{a}_o \approx \frac{a_o}{16} \approx 10^{-2}H$ (81). With reasonable choices for $\beta \approx 100$ and, e.g., $\kappa = 2H^2$ the equation is satisfied, for $\frac{\lambda}{4} \approx 6$.

Although this is a degenerate solution of the W-ST dynamical equations, it is important as a non-homogeneous *point symmetric vacuum solution* with $\nabla\omega = 0$ (respectively with negligible gradient $\nabla\omega \approx 0$). The deviation from the ordinary Schwarzschild equation is only by cosmologically small terms. It thus has the central symmetric point mass solution of the Newton theory as its classical limit. In the next section we see that another classical limit arises as soon as we give up the degeneration condition Einstein gauge = Riemann gauge and we are far away from the source.

4 MOND approximation

4.1 Modified Poisson equation for ω

In the following we assume a weak field constellation in which the Newton approximation of the Einstein equation is justified *even in the presence of a scalar field* ω with purely spacelike variability (signature choice $(-+++)$ of the metric, $\epsilon_{sig} = +1$). This implies a

- (*) condition of *small* acceleration $a_\varphi = -\nabla\omega$, which has to be specified in the particular cases studied.

In the case of exclusively spacelike variability the d'Alembert operator reduces (after sign change) to the Laplacian, $-\square\omega = \Delta\omega$, and the Milgrom operator turns into the known form in spacelike coordinates,

$$\square_M\omega \approx \nabla \cdot (|\nabla\omega|\nabla\omega) \quad (60)$$

(“.” the Euclidean scalar product). For pressure-less matter the Milgrom equation (58) acquires the familiar form of the non-linear Poisson equation³¹

$$\nabla \cdot (|\nabla\omega|\nabla\omega) \approx 4\pi G \tilde{a}_o \rho_m. \quad (61)$$

We call this the *MOND approximation* of W-ST gravity. For the following it is important that only the trace of the *matter* energy momentum tensor, not of the scalar field, appears on the r.h.s. of (61).

Remember that ω is the potential of an additional, not of the total, acceleration, and on the r.h.s of (61) we still have the constant \tilde{a}_o rather than a_o . It will become clear from the growth behaviour of centrally symmetric solutions for ω that we cannot expect the conditions of a W-ST MOND approximation being satisfied in a full spherical neighbourhood of a centrally symmetric mass concentration. $\nabla\omega$ has to be small enough for the energy tensor of the scalar field to be such that its approximative representation in the Newton approximation leads to an acceptable approximation of the Einstein equation (sections 4.3, 4.4). This shows that the MOND approximation may be useful in large distance of a central mass only (if at all). In a closer vicinity the Schwarzschild-de Sitter solution with $\nabla\phi \approx 0$ will be a better approximation (section 3.4) and with it, the Newton approximation, as long as relativistic effects can be neglected.

A simple evaluation shows that for the Newton acceleration a_N ,

$$\nabla^2\Phi_N = 4\pi G \rho_m \quad a_N = -\nabla\Phi_N, \quad (62)$$

the solution of (61) is given by $\nabla\omega = -a_\varphi$ with

$$a_\varphi = \sqrt{\frac{\tilde{a}_o}{|a_N|}} a_N = \sqrt{\tilde{a}_o |a_N|} \frac{a_N}{|a_N|}, \quad (63)$$

³¹ [6, 5].

where $|a_N|$ is the vector norm in the Euclidean approximation.

This is a great relief: The solution of the non-linear Poisson equation is much simpler than one might expect: At first the linear Poisson equation of the Newton theory is to be solved; then an algebraic transformation of type (63) leads to the solution of the non-linear partial differential equation (61).³²

For a point-like mass source M at the origin of spatial coordinates $y = (y_1, y_2, y_3)$, the r.h.s becomes $-\epsilon_{sig} 4\pi G \operatorname{tr} T^{(m)} = 4\pi G M \delta(y)$. Considering an Euclidean approximation for $g_{\mu\nu} \approx \eta_{\mu\nu}$, the corresponding solution is

$$\omega \approx \sqrt{GM\tilde{a}_o} \ln |y|. \quad (64)$$

The Weyl geometric additional acceleration is

$$a_\varphi = -\nabla\omega \approx -\sqrt{GM\tilde{a}_o} \frac{y}{|y|^2}. \quad (65)$$

Its form is the same as the deep MOND acceleration of the usual MOND theory. Then

$$\Delta\omega \approx \frac{\sqrt{GM\tilde{a}_o}}{|y|^2}. \quad (66)$$

The form of (65) shows that the MOND approximation can be reliable only in large distances from the symmetry centre; for ‘small’ radii $(*)$ is no longer satisfied. We shall use the specification $y \geq 10^{-l} \sqrt{GMa_o^{-1}}$. Then $\nabla\omega \leq 4 \cdot 10^l a_o$. With $l = 1$ we are at least in the region called *upper transitional regime* in app. 7.3.³³

4.2 Side remark on the cosmological limiting case

We want to make a short observation with regard to the cosmological limit. If we use the idealizing assumption of homogeneous matter distribution, the invariant scalar field does not depend on the spacelike coordinates of $x = (x_o, x_1, x_2, x_3)$, $x_o = t$,

$$\omega(x) = \omega(t), \quad \nabla\omega = (g^{oo}\omega', 0, 0, 0).$$

We then have a timelike or zero gradient of the scalar field $\nabla\omega$, respectively $D\phi$; $\|\nabla\omega\|$ vanishes, and with it $L_{\phi 3}$ (34) and \square_M . For $L_{\phi 2}$ we have to consider both choices (31, 36).

For (31) with $\alpha = 6$, the scalar field equation reduces to the trace of the vacuum Einstein equation. In other words, it is compatible with the Einstein equation only for $\operatorname{tr} T^{(m)} = 0$, in which case it is redundant. An

³²In the terminology of the MOND community: the MOND approximation of W-St leads to a *QMOND model* [20, pp. 46ff.].

³³“At least”, because further out we enter the MOND regime or even the deep MOND regime of app. 7.3.

inspection of the Friedmann equation (in the Weyl gravity framework) in Einstein gauge shows that the Riemannian component of the scalar curvature must be constant. A special solution is given by the Einstein - de Sitter model with warp function $a(t) = 2\sqrt{\frac{\Lambda}{3}}e^{\pm\sqrt{\frac{\Lambda}{3}}t}$ and vanishing gradient of the scalar field; i.e., the scalar field reduces to the cosmological constant, and Einstein gauge = Riemann gauge.

For (36) both, $L_{\phi 2}$ and $L_{\phi 3}$, vanish in the cosmological case. The scalar field equation reduces to the potential condition (56), and the trace of the Einstein equation to

$${}_g\Box\omega = \frac{4\pi G}{3}tr T^{(m)}. \quad (67)$$

For a Robertson-Walker metric in Einstein gauge

$$g = \epsilon_{sig} \text{diag}(-1, a^2 g_{11}, a^2 g_{22}, a^2 g_{33})$$

with warp function $a = a(t) > 0$, and time independent standard metric of constant curvature on the spacelike slices, $\tilde{g} = \text{diag}(g_{11}, g_{22}, g_{33})$ with. e.g., $g_{11} = (1 - \kappa r^2)^{-1}$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2 \theta$ ($r = x_1, \theta = x_2$), we find

$$\sqrt{|g|} = a^3 \sqrt{|\tilde{g}|}, \quad \frac{\partial_o \sqrt{|g|}}{\sqrt{|g|}} = 3 \frac{a'}{a}.$$

Therefore with (54)

$${}_g\Box\omega = \omega'' + 3 \frac{a'}{a}.$$

and (67) becomes

$$\omega'' + 3 \frac{a'}{a} \omega' = \frac{4\pi G}{3} tr T^{(m)}. \quad (68)$$

For the vacuum case this condition is satisfied by a simple time-homogeneous static solution of the vacuum Einstein equation (46). In Einstein gauge it has the underlying Riemannian geometry of an Einstein universe and a *non-vanishing Weylian scale connection* $\varphi = (H, 0, 0, 0)$ which encodes the cosmological redshift. This implies $\omega = -Ht$, $\omega'' = 0$ and $a' = 0$.³⁴

4.3 Scalar field energy density

We now want to address the distribution of the scalar field's energy density. We use the static weak field approximation (23) in Einstein gauge near a mass center. Then $\omega(x)$ depends only on the spacelike coordinates of $x = (x_o, \dots x_3)$, which we characterize separately by the 3-vector $y := (y_1, y_2, y_3) = (x_1, x_2, x_3)$. The energy-momentum tensor of the scalar field $T^{(\phi)} \doteq (8\pi G)^{-1} \Theta$ is given by (47), (48). Because of $\partial_o \omega = 0$ the second

³⁴[44, 46].

term of the energy density of $\Theta^{(II)}$ vanishes immediately, and the first term in the light of (101).

In the (static) weak field case, the cosmological constant contribution $L_{V4} g_{oo}$ lies many orders of magnitude below energy densities considered here and can be neglected. With $g_{oo} \dot{\approx} \eta_{oo} = -\epsilon_{sig}$ we find

$$\begin{aligned}\Theta_{oo} = \Theta_{oo}^{(I)} &\approx -\phi^{-2} \square \phi^2 - (\xi \phi)^{-2} L_\phi, \\ &\approx -\phi^{-2} \square \phi^2 - \frac{2}{3} (\eta^{-1} \phi)^{-1} \|\nabla \omega\|^3.\end{aligned}\quad (69)$$

With (104) (appendix 7.1) we get

$$\Theta_{oo} \approx -2_g \square \omega - \frac{2}{3} (\eta^{-1} \phi)^{-1} \|\nabla \omega\|^3. \quad (70)$$

In the MOND and (upper) transitional regimes with, say $|a_N| \leq 10^2 a_o$, gives

$$\frac{2}{3} \tilde{a}_o \|\nabla \omega\|^3 \leq 10 a_o^4.$$

It is cosmologically small of order 4 and thus negligible; hence

$$\Theta_{oo} \approx -2_g \square \omega = 2 \Delta \omega.$$

The energy density of the scalar field, ρ_ϕ , in Einstein gauge finally becomes

$$\rho_\phi \dot{\approx} (8\pi G)^{-1} 2 \Delta \omega. \quad (71)$$

4.4 Additional Newton acceleration and determination of \tilde{a}_o

In the Newtonian limit case the energy of the scalar field (71) contributes to the right hand side of the Poisson equation and leads to additional terms Φ_ϕ and a_ϕ of the total Newton potential Φ_{tot} and its acceleration a_{tot}

$$\Phi_{tot} = \Phi_N + \Phi_\phi, \quad a_{tot} = a_N + a_\phi, \quad (72)$$

$$\nabla^2 \Phi_N = 4\pi G \rho_m, \quad \nabla^2 \Phi_\phi = 4\pi G \rho_\phi, \quad (73)$$

where $a_\phi = -\nabla \Phi_\phi$ and a_N like in (62). The Poisson equation for Φ_ϕ and (71) imply

$$a_\phi = -\nabla \Phi_\phi = -\nabla \omega + X = a_\varphi + X, \quad (74)$$

with a vector field X such that $\nabla X = \epsilon_{sig} \Gamma_{jk}^j \partial^k \omega$.

In the central symmetric case (not necessarily with a point-like mass, but with total mass $M(r)$ inside the radius r such that $M'(r) = 0$ at $r = |y| \geq r_o$ for some r_o) (63) implies:

$$\begin{aligned}a_\varphi(y) &= -\frac{\sqrt{\tilde{a}_o G M(r)}}{r} \frac{y}{r} \\ \omega &= \sqrt{\tilde{a}_o G M(r)} \ln r\end{aligned}\quad (75)$$

$$\nabla^2 \omega = \frac{\sqrt{\tilde{a}_o G M(r)}}{r^2} \quad (76)$$

With an Euclidean metric $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\vartheta^2)$ in spherical coordinates (r, θ, ϑ) ,³⁵

$$\epsilon_{sig} \Gamma_{jk}^j \partial^k \omega = \frac{2}{r} \frac{\sqrt{\tilde{a}_o G M(r)}}{r} = 2 \nabla^2 \omega$$

and

$$X = 2 a_\varphi. \quad (77)$$

We finally get an additional acceleration (with regard to the Newton acceleration a_N of ρ_m)

$$a_{add} = a_\varphi + a_\phi = a_\varphi + 3 a_\varphi = 4 a_\varphi, \quad (78)$$

and the total acceleration

$$a = a_N + a_{add} = a_N + 4 a_\varphi$$

With (63)

$$a = a_N \left(1 + \sqrt{\frac{16 \tilde{a}_o}{|a_N|}} \right) \quad (79)$$

Taking (65) into account, the total correction of the original Newton dynamics of a point-like (or point symmetric) source becomes

$$a_{add} = 4 a_\varphi \approx -4 \sqrt{GM \tilde{a}_o} \frac{y}{r^2}. \quad (80)$$

Now we can specify the value of our \tilde{a}_o for which our model gives a total additional acceleration which *in the deep MOND* domain agrees with the acceleration of Milgrom's MOND approach:

$$\tilde{a}_o = \frac{a_o}{16} \approx \frac{H}{100} \approx 8 \cdot 10^{-31} \text{ cm} \leftrightarrow 2 \cdot 10^{-20} \text{ s}^{-1}. \quad (81)$$

Then (80) turns into

$$a_{add} \approx -\sqrt{GM a_o} \frac{y}{r^2}, \quad (82)$$

with the usual MOND acceleration $a_o \approx \frac{H}{6}[c]$, and (79) becomes

$$a = a_N \left(1 + \sqrt{\frac{a_o}{|a_N|}} \right), \quad a_{add} = \sqrt{a_o |a_N|} \frac{a_N}{|a_N|}. \quad (83)$$

The norm of the complete (centrally oriented) radial acceleration in the MOND (and the transitional) regime about a point mass M , or in the case of a point symmetric mass distribution, is given by (norm signs here omitted)

$$a = a_N + a_{add} \approx \frac{GM}{r^2} + \frac{\sqrt{GM a_o}}{r}, \quad (84)$$

³⁵ $\Gamma_{11}^1 = 0, \Gamma_{21}^2 = \Gamma_{31}^3 = r^{-1}.$

and the density of the scalar field halo (71) by

$$\rho_\phi(r) = \frac{3}{4}(4\pi G)^{-1} \frac{\sqrt{GMa_o}}{r^2}. \quad (85)$$

In the case of a points symmetric mass distribution, M has to be read as $M(r)$. We resume: In the domain where the MOND approximation is reliable, the acceleration correction to Newton gravity implied by the W-ST approach with cubic kinetical Lagrangian (for ϕ) consists simply in an *additive term* equal to the *deep MOND acceleration* of the usual MOND approach.

5 Comparison with other MOND models

5.1 Transition function

We can now compare our approach with other MOND models. Simply adding a deep MOND term to the Newton acceleration of a point mass, like in (84), is unusual. It is clear that such an approach does not lead to acceptable results in the 'lower' transitional regime with, say, $a_N > 100 a_o$ (app. 7.3).

M. Milgrom rather considered a multiplicative relation between the MOND acceleration a and the Newton acceleration a_N by a kind of 'dielectric analogy':

$$a_N = \mu\left(\frac{a}{a_o}\right) a, \quad \text{with} \quad \mu(x) \longrightarrow \begin{cases} 1 & \text{for } x \rightarrow \infty \\ x & \text{for } x \rightarrow 0, \end{cases} \quad (86)$$

or the other way round

$$a = \nu\left(\frac{a_N}{a_o}\right) a_N, \quad \text{with} \quad \nu(y) \longrightarrow \begin{cases} 1 & \text{for } y \rightarrow \infty \\ y^{-\frac{1}{2}} & \text{for } y \rightarrow 0. \end{cases} \quad (87)$$

Here $\mu(x) \rightarrow x$ means $\mu(x) - x = \mathcal{O}(x)$, i.e. $\frac{\mu(x)-x}{x}$ remains bounded for $x \rightarrow 0$. From this point of view our acceleration (84) is specified by

$$\mu_w(x) = 1 + \frac{1 - \sqrt{1 + 4x}}{2x} \quad \text{and} \quad \nu_w(y) = 1 + y^{-\frac{1}{2}}. \quad (88)$$

One has to keep in mind that our transition functions μ, ν are only reliable in the MOND and the upper transitional regimes (section 4.1).

This embedding into the MOND family shows that the so-called "Kepler laws of galaxy dynamics" hold for our Weyl geometric scalar tensor (W-ST) model like for all others in the family [20, sec. 5]. But here, different from most other family members, the MOND approximation results from a conceptually (with regard to space-time structure) and physically attractive (comparatively simple Lagrangian) *general relativistic "mother" theory*. Regarding the criteria of naturality and simplicity it may seem superior to the better known relativistic MOND theories TeVeS and Einstein aether theory.

5.2 Scalar field mass and phantom mass

It remains to see how the Weyl geometric MOND model compares with the better studied ones with regard to rotation curves of galaxies, cluster dynamics, and lensing properties. Here we can give only a general overview of such a comparison; a detailed empirical evaluation remains a desideratum.

Equations (78, 83) show that three quarters of the W-ST additive acceleration are due to the scalar field energy density, the *scalar field halo*. That is important because the latter expresses a *true energy density* on the right hand side of the Einstein equation (46) and the Newtonian Poisson equation as its weak field, static limit. It is decisive for *lensing* effects of the additional acceleration. In W-ST we have to distinguish between the influence of the additional structure, scalar field and scale connection, on light rays and on (low velocity) trajectories of mass particles. Bending of light rays is influenced by the scalar field halo only, the acceleration of massive particles with velocities far below c by the the scalar field halo *and* the scale connection.

In the MOND literature the amount of a (hypothetical) mass which in Newton dynamics would produce the same effects as the respective MOND correction a_{add} is called *phantom mass* M_{ph} . In our case, phantom mass and scalar field mass M_ϕ differ:

$$M_{ph} = \frac{4}{3}M_\phi \quad (89)$$

For any member of the MOND family the additional acceleration can be expressed by the modified transition function

$$\tilde{\nu} = \nu - 1 \quad (90)$$

with ν like in (87)

$$a_{add} = \tilde{\nu} \left(\frac{|a_N|}{a_o} \right) a_N. \quad (91)$$

As the potential Φ_{ph} attributed to the the phantom mass density ρ_{ph} satisfies $\nabla^2 \Phi_{ph} = 4\pi G \rho_{ph}$ and $\nabla \Phi_{ph} = -a_{add}$, a short calculation shows that the *phantom mass/energy density* may be expressed as

$$\rho_{ph} = \tilde{\nu} \left(\frac{|a_N|}{a_o} \right) \rho_m - (4\pi G a_o)^{-1} \tilde{\nu}' \left(\frac{|a_N|}{a_o} \right) (\nabla |a_N|) \cdot a_N \quad (92)$$

It consists of a contribution proportional to ρ_m with factor $\tilde{\nu}$, which dominates in regions of ordinary matter, and a term derived from the gradient of $|a_N|$ dominating in the “vacuum” (where however scalar field energy is present). For the Weyl geometric model with $\tilde{\nu}_w(y) = y^{-\frac{1}{2}}$, $\tilde{\nu}'_w(y) = -\frac{1}{2}y^{-\frac{3}{2}}$ this implies:

$$\rho_{ph-w} = \left(\frac{a_o}{|a_N|} \right)^{\frac{1}{2}} \left(\rho_m + (8\pi G)^{-1} \nabla(|a_N|) \cdot \frac{a_N}{|a_N|} \right) \quad (93)$$

$$\rho_\phi = \frac{3}{4} \rho_{ph-w} \quad (94)$$

(94) is another expression for (71). Of course the terminology of “phantom energy” is misleading for ρ_{ph-w} , because three quarters of it are due to the scalar field and thus real rather than phantom.

The total dynamical mass M_{dyn} constituted by a classical mass component (mainly baryonic), here denoted by M_{bar} , and phantom mass differs from the lensing mass M_{lens} :

$$M_{dyn} = M_{bar} + M_{ph} \quad (95)$$

$$M_{lens} = M_{bar} + M_{\phi} = M_{bar} + \frac{3}{4}M_{ph} \quad \text{in W-ST} \quad (96)$$

In our model the lensing mass is *smaller* than the dynamical mass. That looks like bad news for explaining lensing at clusters and microlensing at substructures. But we shall see that the transition function compensates this effect, perhaps even more.

5.3 A first comparison between TeVeS and W-ST

In the TeVeS literature it is taken for granted that its scalar and vector fields, the additional structures of TeVeS, influence light trajectories like a real mass source of the same amount as the phantom mass would do in Einstein gravity [62, secs. 4f.]. Therefore the *dynamical mass* M_{dyn} and the *lensing mass* M_{lens} are identical,³⁶

$$M_{dyn} = M_{lens} = M_{bar} + M_{ph} \quad \text{in TeVeS.} \quad (97)$$

Because of the factor $\frac{3}{4}$ in our (96), lensing effects seem to be stronger in TeVeS than in W-ST. But this inference is not conclusive. Phantom mass calculations depend strongly on the choice of the transition functions $\mu, \nu, \tilde{\nu}$ in the respective MOND model or their TeVeS equivalents.

The W-ST transition function ν_w , respectively $\tilde{\nu}_w$ (88) is larger than the ν -functions usually used in MOND/TeVes: [30, 62] consider

$$\nu_1(y) = \frac{1}{2}(1 + \sqrt{1 + 4y^{-1}})$$

and ν_o corresponding to (Bekenstein’s) $\mu_o(x) = 2x(1 + 2 + \sqrt{1 + 4x})^{-1}$. In his cluster studies R. Sanders uses³⁷

$$\nu_2(y) = \sqrt{\frac{1}{2}(1 + \sqrt{1 + 4y^{-2}})}.$$

The figures 1 and 2 below compare the Weyl geometric function $\frac{3}{4}\tilde{\nu}_w$ (red) governing the density of the scalar field halo with the typical MOND

³⁶[30] seem to doubt the reliability of the MOND approximations in some of the TeVeS calculations in the literature. They develop their own relativistic theory of light bending.

³⁷[40, 41].

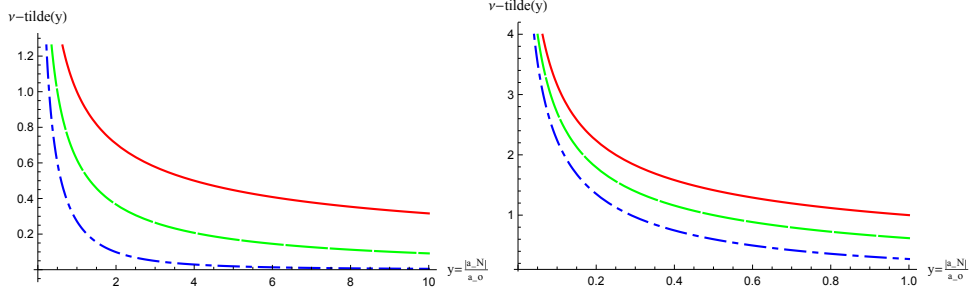


Figure 1: Comparison of phantom halos for Weyl model and usual MOND models (for $\tilde{\nu}$ see (90)). Upper transition regime (left), MOND regime (right); red/unbroken $\tilde{\nu}_w(y)$: indicative of total phantom halo (scalar field and phantom) of Weyl model (see (92)), green/dashed $\tilde{\nu}_1(y)$, blue/double-dashed $\tilde{\nu}_2(y)$ for phantom halos of accepted MOND models.

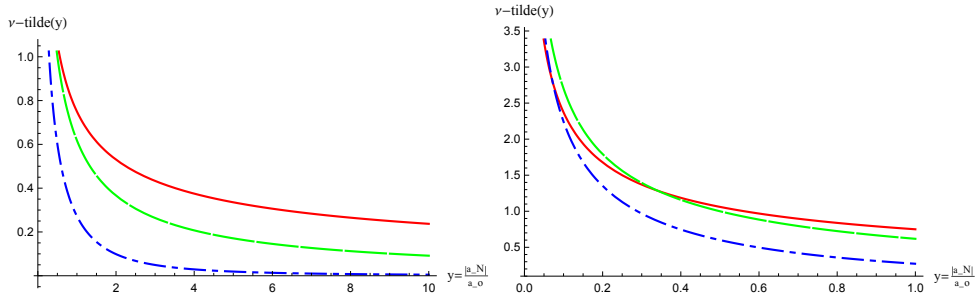


Figure 2: Comparison of scalar field halo for Weyl model with phantom halo of accepted MOND models (for $\tilde{\nu}$ see (90)). Upper transition regime (left) and MOND regime (right); red/unbroken $0.75 \tilde{\nu}_w(y)$: indicative of scalar field halo Weyl model (94), green/dashed $\tilde{\nu}_1(y)$, blue/double-dashed $\tilde{\nu}_2(y)$ for phantom halos of widely used MOND models.

functions $\tilde{\nu}_1$ (green) of Mavromatos e.a. and $\tilde{\nu}_2$ (red) used by Sanders. The $\tilde{\nu}$ -term in (92) dominates the respective phantom energy densities. Figure 1 shows part of the upper transition regime ($1 \leq y \leq 10$ with $y = \frac{|a_N|}{a_o}$) and figure 2 the beginning of the MOND regime ($0.1 \leq y \leq 1$). In the MOND regime ρ_ϕ is close to the phantom energy density of model ν_1 , but much higher than ν_2 . In the transition regime the *W-ST scalar field halo is considerably denser than the phantom energy halo of both received MOND models*. The total phantom energy density of the Weyl approach, which is important for galaxy and cluster dynamics (95), comes out *even higher* and surpasses the phantom energies of the two other models in both domains (figures 3, 4).

These considerations indicate that the missing mass problem for clusters or galaxies, which is being discussed for the MOND-TeVS approach,³⁸ may change its face in the Weyl geometric approach. In the light of the comparison given in figures 3, 4, one might even hope that the mass discrepancy may dissolve under the present dynamical hypothesis.³⁹ But of course this is still far from clear; only detailed empirical studies can show whether the Weyl geometric version of MOND-like weak gravity can really compete with, or even surpass, TeVeS and other relativistic MOND models. In this respect, astronomers will have to speak the final word – if there is any.

6 Discussion

Our assimilation of the original (R)AQUAL Lagrangian to Weyl geometric gravity has shown quite convincing properties. The Weyl geometric approach with its scale covariant expressions is conceptually clearer than the “2-metric-approach” of the Jordan-Brans-Dicke framework in the AQUAL theory. Here Einstein gauge and Riemann gauge, or any other gauge, are mathematically equivalent. Which one seems best depends on the specific problem context. *Einstein gauge* gives the most immediate expression to measured quantities; in this sense it may be considered as the *chronometric gauge*. But it would be misleading to call it *the* “physical gauge”. The affine connection, and with it the gravito-inertial structure is most simply expressed in *Riemann gauge*. Whoever thinks of free fall as being governed by a Levi-Civita connection in the Riemannian sense, may just as well argue for Riemann gauge as “physical”.⁴⁰ A coherent unification of the different aspects of spacetime structure is made possible by a consequently Weyl geometric perspective. The additional degree of freedom (in comparison to Einstein gravity) is related to the new dynamical variable ω . It is regulated by the scalar field equation (58). Because of (16) this equation can also be understood as a condition for the

³⁸See, e.g., [20, 41, 30, 62].

³⁹A model for the halo of galaxy clusters, built on the MOND approximation (which here seems to be of heuristic value only) passes a first empirical surprisingly well [50]

⁴⁰This is reflected in the superiority of Riemann gauge for the variational procedures.

Weylian scale connection. In the degenerate case, $\omega = \text{const}$, the vacuum solution of a point mass source is the Schwarzschild-de Sitter solution with the classical Newtonian limit (section 3.4)

A *first* dynamical consequence of the Weyl geometric extension of Einstein gravity can be identified for low velocity trajectories in the weak field, static approximation in Einstein gauge (the chronometric one). There the Weylian scale connection induces an additional acceleration to the usual Newton approximation of Einstein theory (section 2.5). It has the invariant scalar field ω as its potential (27). It seems quite natural to ask, whether this additional acceleration may be responsible for the anomalous effects of the MOND phenomenology; and if so, under which assumptions for the Lagrangian of the scalar field.

In the *second step* we analyzed whether an adaptation of Bekenstein/Milgrom's non-quadratic Lagrange density for the kinetic term of the scalar field may help to answer this question. Scale invariance gives a strong constraint for the form of the transition function; here it leads to a particularly simple, nearly unique, cubic form (38). In an approximation which allows to apply the Newton approximation of the Einstein equation, the additional acceleration due to the scale connection acquires a *MOND-like* form (section 4). So far our analysis is quite close to RAQUAL, the main differences being scale covariance and the fact that the Newton approximation of Einstein gravity remains a partial contribution of our MOND approximation ((22), (26)).

In a *third step* we have analyzed the energy density of the scalar field (sections 4.3 and 4.4) and found that it *modifies the total Newton potential* of the static weak field approximation (71), (72), (73). That is a result of analyzing the r.h.s of the scale invariant Einstein equation; it needs no additional stipulation. If compared with the original RAQUAL approach, this consequence of our approach changes the situation for *gravitational lensing* and for cluster dynamics considerably.

Given that the last mentioned problems (cluster dynamics and gravitational lensing) seem to have been most decisive for giving up the original RAQUAL approach, one may ask why a similar observation has not been made already long since. The answer seems to reside in a widely spread conviction that scale covariant (or conformal) metrical approaches can never lead to a derivation of gravitational lensing effects. This conviction seems to have acquired the status of a kind of "folk theorem".⁴¹

This conviction has a true core, but it does not express the whole story. Like Diogenes who proved the possibility of motion to the Eleatic critics by walking, we have shown that there *is* an alternative. It is not difficult to see why it could work. The folk theorem has a premiss which often remains unstated. In the following quote it is stated explicitly:

"... so long as the ψ field [corresponding to our ω , E.S.] con-

⁴¹See, among others, [42, pp. 146f.]

tributes comparably little to the energy-momentum tensor, it cannot affect light deflection ...” [5, p. 6, emph. E.S.].

Why does this condition not apply to our Weyl geometric extension of essentially the same Lagrangian like in RAQUAL?

The answer can be read off from (47) and (44). The crucial difference in our energy-momentum tensor to the one often used in JBD-approaches,⁴² comes from the *boundary terms* arising during the *variation of the Hilbert action* to which the scalar field is non-minimally coupled.⁴³ Among these terms, it is mainly $D_\nu D^\nu \phi^2$ which contributes essentially to the energy-momentum (69). The successful adaptation of a cubic scalar field Lagrangian to Weyl geometric gravity is a strong sign for the importance of the boundary terms.

It is too early to draw full consequences of this analysis at the moment. We still have to see whether the Weyl geometric approach proves to be of *empirical relevance* for extremely weak field domains at galaxy and perhaps even at galaxy cluster level, and whether a further analysis of domains, in which neither the MOND approximation nor the Schwarzschild-de Sitter approximation can be applied, sheds new *theoretical* light on strong field constellations. In the case of positive answers, or at least one with encouraging result, we may conclude that the energy density of the gravitational scalar field analyzed in our approach is *real* and not just a model artefact. First indications that the chances for a positive outcome of the empirical examination of our model are not bad are given section 5.3.

If so, we may interpret (47), (48), and (69) as expressions for the *energy* of the (Weyl geometrically) enhanced *gravitational field*. Sceptics ought to remember that a complete spacetime structure is given by the combination of a causal structure (mathematically a conformal structure), an inertio-gravitational structure (projective path structure), and the scalar field specifying the remaining chronometric scaling degree of freedom, mathematically by the triple $(\mathfrak{c}, \nabla, \phi)$ (section 2.4). *Gravitation* is a *complex structure*, not just one (vector, tensor, or connection) field.

This insight may also become important for quantum gravity: In which sense could it be meaningful to quantize the *basic geometrical* features of spacetime, i.e. the conformal and affine structures (\mathfrak{c}, ∇) ? It is well known that these structures do not carry intrinsic, covariant self energy, while the scalar chronometric field does! This speaks in favour of focussing the quantization of gravity, at least in a first step, on the chronometric/scale degree of freedom ϕ and to analyze how the latter relates to the quantized standard model fields on general relativistic spacetime.

⁴²Although some of the JBD literature does take account of the boundary terms of partial integration, e.g., [23, pp. 40ff.].

⁴³See the literature in fn. 29.

7 Appendices

7.1 Scale invariant version of scalar field

In Riemann gauge $(\tilde{g}, 0, \tilde{\phi})$ we write $\tilde{\phi} = e^\omega$ (ω stands here for the scale invariant form). By definition ω is not affected by regauging, therefore

$$D_\nu \omega = \partial_\nu \omega. \quad (98)$$

It is a *scale invariant* version of the *scalar field*.

Any scale gauge (g, φ, ϕ) arises from Riemann gauge, $g = \Omega^2 \tilde{g}$, for some Ω . Then

$$\varphi = -d \ln \Omega \leftrightarrow \Omega = e^{-\int \varphi};$$

here $\int \varphi$ is an abbreviated notation for integrating the 1-form φ along any curve from a fixed initial point to the point x of spacetime considered (underdetermination only up to a point independent constant). We thus get

$$\begin{aligned} \tilde{\phi} &= \Omega \phi, \\ \omega &= \ln \tilde{\phi} = \ln \phi - \int \varphi, \\ \phi &= \Omega^{-1} e^\omega = e^{\omega + \int \varphi}. \end{aligned} \quad (99)$$

In some of the recent literature $\phi_{comp} := e^{\int \varphi}$ is considered on its own (with $\omega = 0$) [2, 3]. It is a “compensating field” for the effects of a conformal transformation away from Riemann gauge. Because of the gauge transformation for the scale connection it transforms with weight $w(\phi_{comp}) = -1$ like ϕ . But it does not essentially contribute to the dynamics besides giving it a scale covariant expression. Restricting to ϕ_{comp} boils down to considering Einstein gravity in scale covariant form. The result is a *dynamically trivial* Weyl geometric extension of Einstein gravity (and Riemannian geometry).

If (g, φ, ϕ_o) denotes a scalar field gauge, in particular Einstein gauge $\phi_o \doteq \xi^{-1} E_{pl}$, we have $\phi_o = \Omega^{-1} \tilde{\phi}$ with $\Omega = \phi_o^{-1} e^\omega = \xi E_{pl}^{-1} e^\omega$; thus $\varphi \doteq -d \ln \Omega \doteq -d\omega$ and

$$\varphi_\nu \doteq -\partial_\nu \omega. \quad (100)$$

Thus ω has the formal properties of a potential for the scale connection φ in *scalar field gauge* (and only in this gauge).

The scale covariant derivative of the scalar field in any gauge can be expressed as follows:

$$\begin{aligned} D_\nu \phi &= (\partial_\nu - \varphi_\nu) \phi = \partial_\nu e^{\omega + \int \varphi} - \varphi_\nu \phi = (\partial_\nu \omega + \varphi_\nu) \phi - \varphi_\nu \phi \\ &= \phi \partial_\nu \omega = \phi D_\nu \omega \end{aligned} \quad (101)$$

Similarly one derives

$$D^\nu \phi^2 = 2\phi^2 \partial^\nu \omega = 2\phi^2 D^\nu \omega \quad (102)$$

and

$$\begin{aligned}
D_\nu D^\nu \phi^2 &= D_\nu (2\phi^2 D^\nu \omega) \\
&= 2\phi^2 (D_\nu D^\nu \omega + 2D_\nu \omega D^\nu \omega) \\
&= 2\phi^2 (\nabla_\nu \partial^\nu \omega - 2\varphi_\nu \partial^\nu \omega + 2\partial_\nu \omega \partial^\nu \omega)
\end{aligned}$$

Because of $\varphi \Gamma_{\nu\mu}^\lambda = \delta_\nu^\lambda \varphi_\mu + \delta_\mu^\lambda \varphi_\nu - g_{\nu\mu} \varphi^\lambda$ we find

$$\begin{aligned}
\nabla_\nu \partial^\nu \omega &= g \nabla_\nu \partial^\nu \omega + \varphi \Gamma_{\nu\mu}^\nu \partial^\mu \omega \\
&= g \nabla_\nu \partial^\nu \omega + 4\varphi_\nu \partial^\nu \omega
\end{aligned}$$

In scalar field gauge $\varphi \doteq -\partial_\nu \omega$ and thus

$$D_\nu D^\nu \phi^2 \doteq 2\phi^2 g \nabla_\nu \partial^\nu \omega. \quad (103)$$

In terms of the signature normalized Beltrami-d'Alembert operators (53)

$$\square \phi^2 \doteq 2\phi^2 g \square \omega. \quad (104)$$

7.2 Derivation of the scalar field equation

We use scale covariant variation $\delta\omega$ with regard to the scale invariant scalar field ω as dynamical variable, observing that $D_\nu \omega = \partial_\nu \omega$.⁴⁴ We calculate the variation in Riemann gauge (then R contains no φ -terms). Because of scale invariance of the Lagrangian, the result translates straight forward to any gauge. The Lagrange density (37) is constructed using scale covariant derivatives D_ν . The appropriate Euler-Lagrange equation is $\frac{\delta L_\phi}{\delta \phi} = \frac{\partial L_\phi}{\partial \phi} - D_\nu \frac{\partial L_\phi}{\partial (\partial_\nu \phi)}$ [22, p. 526].

Using (50) we get:

$$\begin{aligned}
\frac{\delta L_{HW}}{\delta \omega} &= \frac{\partial L_{HW}}{\partial \omega} = \epsilon_{sig} \phi^2 R = 2L_{HW} \\
\frac{\delta L_{V4}}{\delta \omega} &= \frac{\partial L_{V4}}{\partial \omega} = 4L_{V4} \\
\frac{\delta L_{\phi 2}}{\delta \omega} &= \frac{\partial L_{\phi 2}}{\partial \omega} = 2L_{\phi 2}, \quad \frac{\delta L_{\phi 3}}{\delta \omega} = \frac{\partial L_{\phi 3}}{\partial \omega} = L_{\phi 3}
\end{aligned}$$

Moreover,

$$\frac{\partial L_{\phi 2}}{\partial (\partial_\nu \omega)} = \epsilon_{sig} \alpha (\xi \phi)^2 \partial^\nu \omega,$$

and with (51)

$$\frac{\partial L_{\phi 3}}{\partial (\partial_\nu \omega)} = 2\epsilon_{sig} \xi^2 \eta \phi \|\nabla \omega\| \partial^\nu \omega$$

⁴⁴Equivalently, variation with regard to ϕ could be taken.

for $\nabla\omega$ (respectively $D\phi$) spacelike (otherwise zero). Because of

$$\begin{aligned} D_\nu \frac{\partial L_{\phi 3}}{\partial(\partial_\nu \omega)} &= 2\epsilon_{sig} \xi^2 \eta D_\nu (\phi \|\nabla\omega\| \partial^\nu \omega) \\ &= 3L_{\phi 3} + 2\xi^2 \eta \phi D_\nu (\epsilon_{sig} \|\nabla\omega\| \partial^\nu \omega) \\ &= 3L_\phi + 2\xi^2 \eta \phi \square_M \omega \quad (\text{cf. (52)}) \end{aligned}$$

and $D_\nu \frac{\partial L_{\phi 2}}{\partial(\partial_\nu \omega)} = 2L_{\phi 2} + \epsilon_{sig} \alpha \xi^2 \phi D_\nu D^\nu \phi$, this leads to the “raw” scalar field equation

$$2L_{HW} + 4L_{V4} - 2L_{\phi 3} + \alpha \xi^2 \phi \square \phi - 2(\xi \phi)^2 (\eta^{-1} \phi)^{-1} \square_M \omega = 0 \quad (105)$$

for $D\phi$ spacelike. Otherwise, i.e. $D\phi$ causal, the $L_{\phi 3}$ and $\square_M \omega$ terms vanish; for the choice (36) also the term in $\square \phi$.

On the other hand, tracing of the Einstein equation (46) and multiplication by $\epsilon_{sig}(\xi \phi)^2$ leads to:

$$2L_{HW} + 4L_{V4} + 2\left(1 - \frac{6}{\alpha}\right)L_{\phi 2} + L_{\phi 3} + 6\phi \xi^2 \square \phi + \epsilon_{sig} \text{tr} T^{(m)} = 0 \quad (106)$$

For $\alpha = 6$ and spacelike $D\phi$ (respectively spacelike $\nabla\omega$) the subtraction of (106) from equ. (105) leads to the simplified scalar field equation (55), of the main text:

$$2(\xi \phi)^{-2} (\eta^{-1} \phi) \square_M \omega = -\epsilon_{sig} \text{tr} T^{(m)} - 3L_{\phi 3}$$

Without the $L_{\phi 2}$ term and $\alpha = 6$ additional terms (in $\square \phi$ and proportional to $L_{\phi 2}$) would appear.⁴⁵ On shell of the Einstein equation (58) is equivalent to the raw scalar equation. For causal $D\phi$ and assuming $\alpha = 6$, the scalar field equation is consistent with the Einstein equation only for $\text{tr} T^{(m)} = 0$.

7.3 MOND, deep MOND, and transition regimes

A point is called to lie in the *MOND regime*, if the Newton acceleration falls below a_o : $a_N \leq a_o$ (here a_N, a_{add} denote the norm of the accelerations). In our approach with additional acceleration $a_{add} = \sqrt{a_N a_o}$ (83) this is equivalent to $a_{add} \geq a_N$.

If we agree to speak of *deep MOND regime* (dM), if the additional acceleration strongly dominates the Newton acceleration, $a_{add} \gg a_N$ in the sense of, say, $a_{add} \geq 10 a_N$ (or $a_{add} \geq 10^l a_N$), the dM condition is equivalent to $a_N \leq 10^{-2} a_o$ (respectively $a_N \leq 10^{-2l} a_o$).

For $a_o \leq a_N \leq 100a_o$ we speak of the *upper transition regime* from Newton to MOND. For the ‘lower’ transition regime with $a_N > 100a_o$ the MOND approximation of W-ST loses its reliability (section 4.1).

⁴⁵They are called “nuisance terms” in the published version of this paper and in earlier preprints.

In short: we have dM for $\frac{a_N}{a_o} \leq 10^{-2}$, MOND regime for $\frac{a_N}{a_o} \in [0.01, 1]$, and the upper transition regime if $\frac{a_N}{a_o} \in [1, 100]$ (for $k = l = 1$). For a central symmetric mass M the MOND regime starts at the distance $r_o = \sqrt{GMa_o^{-1}}$, the transition regime at $10^{-1} r_o$, dM at $10 r_o$.

For stars with size of the sun, $GM_\odot \sim 10^5 \text{ cm}$, and with $a_o \sim \frac{H}{6} \sim 10^{-29} \text{ cm}^{-1}$ we get $r_o \sim 10^{17} \text{ cm} \sim 10^4 \text{ AU} \sim 10^{-1} \text{ pc}$. For the mass of a galaxy with $M_{gal} \sim 10^{11} M_\odot$, idealized to spherical symmetry, the MOND regime of the total galaxy begins 5 to 6 orders of magnitude higher, $r_o \sim 10 \text{ kpc}$, the deep MOND at the outskirts of the disk $R_1 \sim 100 \text{ kpc}$. Note that the stars constituting the galaxy have their own MOND and dM regimes at the lower scale. In our approach, their scalar field halos contribute to the total gravitational mass-energy of the galaxy and are crucial for microlensing effects.

References

- [1] Adler, Ronald; Bazin, Maurice; Schiffer Menahem. 1975. *Introduction to General Relativity*. New York etc.: Mc-Graw-Hill. 2nd edition.
- [2] Almeida, T.S.; Formiga, J.B. Pucheu Maria L.; Romero C. 2014a. "From Brans-Dicke gravity to a geometrical scalar-tensor theory." *Physical Review D* 89:064047 (10pp.). arXiv:1311.5459.
- [3] Almeida, T.S.; Pucheu, M.L.; Romero C. 2014b. A geometrical approach to Brans-Dicke theory. In *Accelerated Cosmic Expansion. Proceedings of the Fourth International Meeting On Gravitation and Cosmology*, ed. L.M. Reyes Barrera C. Moreno Gonzales, J. E. Madriz Aguilar. Heidelberg etc.: Springer pp. 33–42.
- [4] Audretsch, Jürgen; Gähler, Franz; Straumann Norbert. 1984. "Wave fields in Weyl spaces and conditions for the existence of a preferred pseudo-riemannian structure." *Communications in Mathematical Physics* 95:41–51.
- [5] Bekenstein, Jacob. 2004. "Relativistic gravitation theory for the modified Newtonian dynamics paradigm." *Physical Review D* 70(083509).
- [6] Bekenstein, Jacob; Milgrom, Mordechai. 1984. "Does the missing mass problem signal the breakdown of Newtonian gravity?" *Astrophysical Journal* 286:7–14.
- [7] Blagojević, Milutin. 2002. *Gravitation and Gauge Symmetries*. Bristol/Philadelphia: Institute of Physics Publishing.
- [8] Bureau International des poids et mesures. 2011. "Resolutions adopted by the General Conference on Weights and Measures (24th meeting), Paris, 17–21 October 2011." www.bipm.org/en/si/new_si/.
- [9] Calderbank, D; Pedersen, H. 2000. Einstein-Weyl geometry. In *Surveys in Differential Geometry. Essays on Einstein Manifolds*, ed. C. Le Brun; M. Wang. Boston: International Press pp. 387–423.

- [10] Callan, Curtis; Coleman, Sidney; Jackiw Roman. 1970. “A new improved energy-momentum tensor.” *Annals of Physics* 59:42–73.
- [11] Carroll, Robert. 2004. “Gravity and the quantum potential.” Preprint. arXiv:gr-qc/0406004.
- [12] Cheng, Hung. 1988. “Possible existence of Weyl’s vector meson.” *Physical Review Letters* 61:2182–2184.
- [13] De Martini, Francesco; Santamato, Enrico. 2013. “Derivation of Dirac equation by conformal differential geometry.” *Foundations of Physics* 43(5):631–641. arXiv:1107.3168.
- [14] De Martini, Francesco; Santamato, Enrico. 2014. “Interpretation of the quantum-nonlocality by conformal geometrodynamics.” *International Journal of Theoretical Physics* 53: 3308–3322. arXiv:1203.0033.
- [15] Di Mauro, M.; Fatibene, L.; Ferraris M.; Francaviglia M. 2010. “Further Extended Theories of Gravitation.” *International Journal of Geometrical Methods in Modern Physics* 7(5):887–898. arXiv:0911.2841.
- [16] Dicke, Robert H. 1962. “Mach’s principle and invariance under transformations of units.” *Physical Review* 125:2163–2167.
- [17] Dirac, Paul A.M. 1973. “Long range forces and broken symmetries.” *Proceedings Royal Society London A* 333:403–418.
- [18] Drechsler, Wolfgang; Tann, Hanno. 1999. “Broken Weyl invariance and the origin of mass.” *Foundations of Physics* 29(7):1023–1064. arXiv:gr-qc/98020.
- [19] Ehlers, Jürgen; Pirani, Felix; Schild Alfred. 1972. The geometry of free fall and light propagation. In *General Relativity, Papers in Honour of J.L. Synge*, ed. L. O’Raifeartaigh. Oxford: Clarendon Press pp. 63–84.
- [20] Famaey, Benoît; McGaugh, Stacy. 2012. “Modified Newtonian dynamics (MOND): Observational phenomenology and relativistic extensions.” *Living Reviews in Relativity* 15(10):1–159.
- [21] Folland, George B. 1970. “Weyl manifolds.” *Journal of Differential Geometry* 4:145–153.
- [22] Frankel, Theodore. 1997. *The Geometry of Physics*. Cambridge: University Press. 2nd ed. 2004.
- [23] Fujii, Yasunori; Maeda, Kei-Chi. 2003. *The Scalar-Tensor Theory of Gravitation*. Cambridge: University Press.
- [24] Gilkey, Peter; Nikčević, Stana; Simon, Udo. 2011. “Geometric realizations, curvature decompositions, and Weyl manifolds.” *Journal of Geometry and Physics* 61:270–275. arXiv:1002.5027.
- [25] Hayashi, Kenji; Kugo, Taichiro. 1979. “Remarks on Weyl’s gauge field.” *Progress of Theoretical Physics* 61:334–346.

- [26] Hehl, Friedrich W.; McCrea, J. Dermott; Mielke Eckehard; Ne’eman Yuval. 1989. “Progress in metric-affine gauge theories of gravity with local scale invariance.” *Foundations of Physics* 19:1075–1100.
- [27] Higa, Tatsuo. 1993. “Weyl manifolds and Einstein-Weyl manifolds.” *Commentarii Mathematici Sancti Pauli* 42:143–160.
- [28] Kroupa, Pavel; Milgrom, Mordechai; Pawłowski Marcel. 2012. “The failures of the standard model of cosmology require a new paradigm.” *International Journal of Modern Physics D* 21(14):120003–1–13.
- [29] Mannheim, Philip. 2006. “Alternatives to dark matter and dark energy.” *Progress in Particle and Nuclear Physics* 56:340f–445. arXiv:astro-ph/0505266.
- [30] Mavromatos, Nick; Sakellariadou, Mairi; Furqaan Yusa Muhammad. 2009. “Can TeVeS avoid Dark Matter on galactic scales?” *Physical Review D* 79:081301. arXiv:0901.3932.
- [31] Omote, M. 1971. “Scale transformations of the second kind and the Weyl space-time.” *Lettere al Nuovo Cimento* 2(2):58–60.
- [32] Omote, M. 1974. “Remarks on the local-scale-invariant gravitational theory.” *Lettere al Nuovo Cimento* 10(2):33–37.
- [33] O’Raifeartaigh, Lochlainn. 1997. *The Dawning of Gauge Theory*. Princeton: University Press.
- [34] Ornea, Liviu. 2001. “Weyl structures on quaternionic manifolds. A state of the art.” Preprint. arXiv:math/0105041.
- [35] Poulis, Felipe P.; Salim, J.M. 2011. “Weyl geometry as a characterization of space-time.” *International Journal of Modern Physics: Conference Series V* 3:87–97. arXiv:1106.3031.
- [36] Quiros, Israel. 2013. “Scale invariance and broken electroweak symmetry may coexist together.” Preprint. arXiv:1312.1018.
- [37] Quiros, Israel. 2014. “Scale invariant theory of gravity and the standard model of particles.” Preprint Guadalajara. arXiv:1401.2643.
- [38] Quiros, Israel; García-Salcedo, Ricardo; Madriz Aguilar José E.; Matos Tonatiuh. 2013. “The conformal transformations’ controversy: what are we missing.” *General Relativity and Gravitation* 45:489–518. arXiv:1108.5857.
- [39] Romero, Carlos, Fonseca-Neto J.B.; Pucheu Maria L. 2011. “General relativity and Weyl frames.” *International Journal of Modern Physics A* 26(22):3721–3729. arXiv:1106.5543.
- [40] Sanders, Robert. 1999. “The virial discrepancy in clusters of galaxies in the context of modified Newtonian dynamics.” *Astrophysical Journal* 512:L23–L26.
- [41] Sanders, Robert. 2003. “Clusters of galaxies with modified Newtonian dynamics.” *Monthly Notices Royal Astronomical Society* 342:901–908.

- [42] Sanders, Robert H. 2010. *The Dark Matter Problem. A Historical Perspective*. Cambridge: University Press.
- [43] Santamato, E. 1984. “Geometric derivation of the Schrödinger equation from classical mechanics in curved Weyl spaces.” *Physical Review D* 29:216–222.
- [44] Scholz, Erhard. 2005*a*. Einstein-Weyl models of cosmology. In *Albert Einstein. 100 Authors for Einstein*, ed. J. Renn. Weinheim: Wiley-VCH pp. 394–397.
- [45] Scholz, Erhard. 2005*b*. “On the geometry of cosmological model building.” Preprint. arXiv:gr-qc/0511113.
- [46] Scholz, Erhard. 2009. “Cosmological spacetimes balanced by a Weyl geometric scale covariant scalar field.” *Foundations of Physics* 39:45–72. arXiv.org/0805.2557.
- [47] Scholz, Erhard. 2011. “Weyl geometric gravity and electroweak symmetry ‘breaking’.” *Annalen der Physik* 523:507–530. arxiv.org:1102.3478.
- [48] Scholz, Erhard. 2014. Paving the way for transitions – a case for Weyl geometry. To appear in *Towards a Theory of Spacetime Theories*, ed. D. Lehmkuhl e.a. Basel: Birkhäuser (Springer). arXiv:1206.1559.
- [49] Scholz, Erhard. 2015. “Higgs and gravitational scalar fields together induce Weyl gauge.” *General Relativity and Gravitation* 47(7). arXiv:1407.6811.
- [50] Scholz, Erhard. 2015. “A new halo model for clusters of galaxies.” *Preprint*. arXiv:1506.09138
- [51] Shojai, Fatimah; Shojai, Ali. 2002. “Weyl geometry and quantum gravity.” Preprint AEI-2002-060. gr-qc/0306099.
- [52] Smolin, Lee. 1979. “Towards a theory of spacetime structure at very short distances.” *Nuclear Physics B* 160:253–268.
- [53] Stachel, John. 2003. A brief history of space-time. In *A Relativistic Spacetime Odyssey. Experiments and Theoretic Viewpoints onf General Relativity and Quantum Gravity*, ed. I. Ciufolini e.a. Singapore etc.: World Scientific pp. 15–34.
- [54] Starkman, Glenn. 2011. “Modifying gravity: You can’t always get what you want.” *Philosophical Transactions Royal Society* 369(A28):5018–5041. arXiv:1201.1697.
- [55] Tann, Hanno. 1998. *Einbettung der Quantentheorie eines Skalarfeldes in eine Weyl Geometrie — Weyl Symmetrie und ihre Brechung*. München: Utz.
- [56] ’t Hooft, Gerard. 2014. “Local conformal symmetry: The missing symmetry component for space and time.” *Preprint* Essay written for the Gravity Research Foundation 2015. arXiv:1410.667
- [57] Utiyama, Ryoyu. 1975*a*. “On Weyl’s gauge field.” *General Relativity and Gravitation* 6:41–47.

- [58] Utiyama, Ryoyu. 1975*b*. “On Weyl’s gauge field II.” *Progress of Theoretical Physics* 53:565–574.
- [59] Weinberg, Stephen. 1972. *Gravitation and Cosmology*. New York: Wiley.
- [60] Weyl, Hermann. 1918. “Gravitation und Elektrizität.” *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin* pp. 465–480. In [61, II, 29–42] [31], English in [33, 24–37].
- [61] Weyl, Hermann. 1968. *Gesammelte Abhandlungen, 4 vols.* Ed. K. Chandrasekharan. Berlin etc.: Springer.
- [62] Zhao, Hongshen; Bacon, David; Taylor Andy; Horne Keith. 2006. “Testing Bekenstein’s relativistic modified Newtonian dynamics with lensing data.” *Monthly Notices Royal Astronomical Society* 368:171–186.